has been found to be 5. Values of the coefficients a_i (are shown in Table I. The following transformation formulas are helpful in restricting K/K' to the interval (0,1)

$$K(k') = K'(k) \qquad K(k) = K'(k') \qquad k^2 + k'^2 = 1. \quad (3)$$

Equation (2) is valuable in practice, since it allows the synthesis of all transmission lines to be realized. A few of them are shown in Fig. 1.

In fact, for a given characteristic impedance, the corresponding geometric dimensions of the line can be calculated rapidly by means of an electronic pocket calculator. Moreover, since the two developments are valid for all values of the characteristic impedance, one can know immediately if the realization of the latter is possible or not for a chosen geometrical configuration.

Hence, provided that the dielectric interfaces of geometrical configurations do not present any difficulty in conformal mapping, analytical expressions obtained can be easily manipulated, without having to resort to tables.

We hope that these synthesis formulas will find a place in the bibliography and allow engineers to make use of elliptic integrals with less hesitation.

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A Two- or Three-Dimensional Green's Function Which Can Be Applied to Hyperfrequency Microelectronic Transmission Lines

RAYMOND CRAMPAGNE AND JEAN-LOUIS GUIRAUD

Knowing Green's function and the charge density found on different conductors, the diverse capacities [1]–[4] can eventually be calculated by solving an integral equation. This has been dealt with only for simple dielectric-conductor configurations. In Coen's article [5], the integral representation of log (Z) is employed in calculating Green's function for microstrips (with or without an upper ground plane). Electrostatically speaking, the boundary conditions along conductors or dielectric interfaces are represented by means of infinite charge series.

We will treat two- or three-dimensional problems in exactly the same way; microstrip [Fig. 1(b) and (c)], triplate [Fig. 1(a)], and coplanar [Fig. 1(d) and (e)] types of transmission lines in a quasi-TEM approximation can be treated in the two-dimensional case [1], [2]. As indicated in Fig. 1, these lines can also be composed of several dielectrics. The three-dimensional case is employed in the calculation of capacitances or inductances of equivalent circuits representing discontinuities of certain lines [Fig. 1(*f*)] or the capacitances obtained by the localized element technique [Fig. 1(g)].

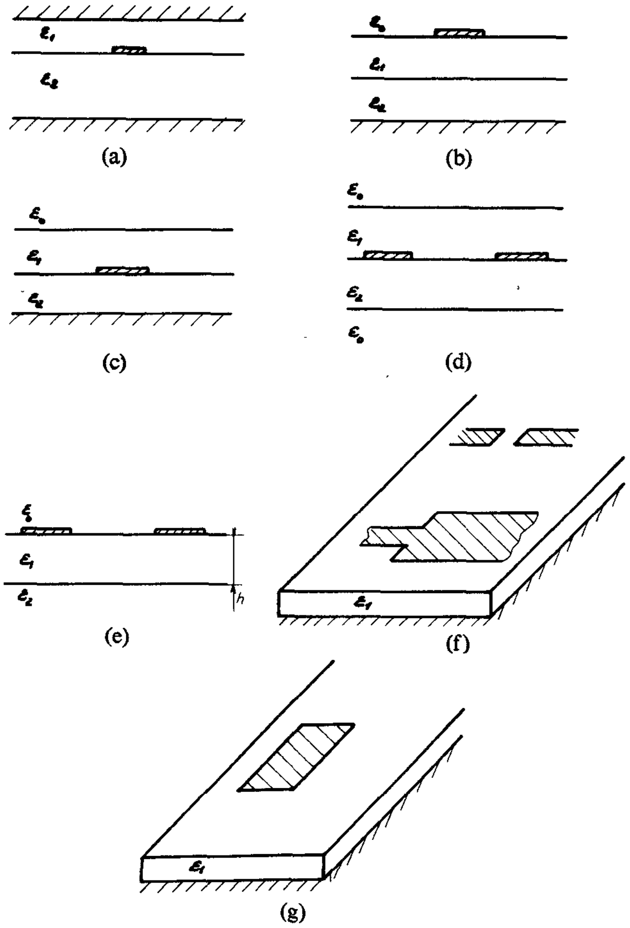

Fig. 1.

The aim is to find an integral representation of Green's functions in space with several dielectrics: log (Z) or $1/r$ depending upon whether the Green's functions in free space are a two- or three-dimensional problem. $Z = y + jx$ is a point in the Z plane which represents the cross section of a line charge; $r^2 = \rho^2 + u^2$ represents the distance between the point field and the point source of a point charge.

In the case of a homogeneous dielectric body of permittivity ε, the integral representation of Green's function for a line charge [5] situated at $x = 0$, $y = \alpha$ or a point charge [6] situated at $\rho = 0$, $u = \alpha$ can be written as

$$\phi(x,y) = \frac{1}{2\pi\varepsilon} \int_0^\infty \frac{\exp[-\lambda|y - \alpha|] \cos \lambda x - \exp[-\lambda]}{\lambda} \, d\lambda$$

or

$$\phi(\rho,u) = \frac{1}{4\pi\varepsilon} \int_0^\infty J_0(\lambda\rho) \exp[-\lambda|u - \alpha|] \, d\lambda. \quad (1)$$

All further developments will be based upon the following remark: A multiplication within the integral of expressions [1]

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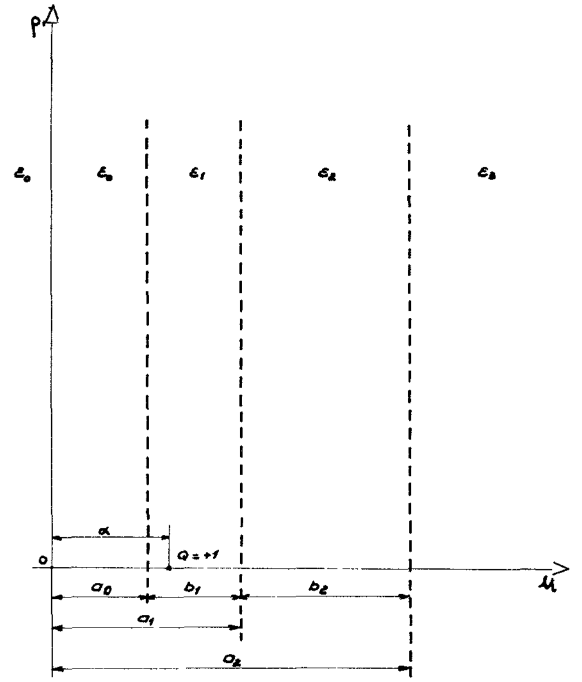

Fig. 2. Point charge in an inhomogeneous medium of four dielectrics.

by any function $f_i(\lambda)$ of the variable λ will always give rise to a potential function which satisfies Poisson's equation.

The geometric configuration treated here consists of four dielectrics and a point or line charge, depending on the case to be considered (Fig. 2). The choice is clear: The geometrical configurations of Fig. 1(a)–(g) are only special cases. Electrostatically speaking, a perfect conductor can be simulated by imposing a dielectric constant of infinite value to the medium.

Therefore, in each dielectric medium characterized by an index i (i = 0–3), we will try to express Green's function in the following form:

Line Charge

$$f_i(\lambda) \equiv F_i^{\pm}(\lambda)$$

$$\phi_i = \frac{1}{2\pi\varepsilon_1}\int_0^{\infty} \frac{\{F_i^-(\lambda)\exp[-\lambda y] + F_i^+(\lambda)\exp[+\lambda y] + \exp[-\lambda|y-\alpha|]\delta_{i,1}\}\cos\lambda x}{\lambda}\,d\lambda. \tag{2}$$

Point Charge

$$f_i(\lambda) \equiv G_i^{\pm}(\lambda)$$

$$\phi_i = \frac{1}{4\pi\varepsilon_1}\int_0^{\infty}\{G_i^-(\lambda)\exp[-\lambda u] + G_i^+(\lambda)\exp[+\lambda u] + \exp[-\lambda|u-\alpha|]\delta_{i,1}\}J_0(\lambda\rho)\,d\lambda$$

$$\delta_{i,1} = \begin{cases}1, & \text{for } i = 1\\ 0, & \text{for } i \neq 1.\end{cases}$$

The term $\delta_{i,1}$ signifies that the only charge having a physical meaning is found in the medium of dielectric constant ε_1. Note that it is sufficient to know the potential of the medium 1 in order to characterize the charge density of conductors [Fig. 1(a)–(g)].

The functions $F_i^{\pm}(\lambda)$ and $G_i^{\pm}(\lambda)$ (i = 0–3) are determined by writing the condition at infinity and the boundary conditions along the dielectric interfaces: continuity of the potential and the normal derivatives of the electric field. Since the initial equations are strictly identical, the calculations lead to identical expressions for the functions $F_i^{\pm}(\lambda)$ and $G_i^{\pm}(\lambda)$, the conditions along dielectric interfaces being satisfied for all values of x or ρ. Thus

$$F_i^{\pm}(\lambda) \equiv G_i^{\pm}(\lambda) \qquad \forall i.$$

The convergence of the integrals (2) can be verified after the $f_i(\lambda)$ are found.

In order to simplify notations, let

$$\varepsilon_i^* = \frac{\varepsilon_i}{\varepsilon_{i-1}} \quad k_i = \frac{\varepsilon_i^* - 1}{\varepsilon_i^* + 1} \qquad \forall i,\ i = 1,2,3$$

$$F(k_1,k_2,k_3) = \frac{1}{1 + FF}$$

with

$$FF = k_1k_2\exp[-2\lambda b_1] + k_2k_3\exp[-2\lambda b_2] + k_1k_3\exp[-2\lambda(b_1 + b_2)]. \tag{3}$$

Expressions of the functions $F_i^{\pm}(\lambda)$ can then be written as

$$F_1^-(\lambda) = -k_1\exp[\lambda(\alpha - 2b_1)]\{k_2 + k_3\exp[-2\lambda b_2]\} \cdot \{1 + k_1\exp[-2\lambda(\alpha - a_0)]\}F(k_1,k_2,k_3) + k_1\exp[\lambda(\alpha - 2(\alpha - a_0))] \tag{4}$$

$$F_1^+(\lambda) = -\exp[\lambda(-2a_1 + \alpha)]\{k_2 + k_3\exp[-2\lambda b_2]\} \cdot \{1 + k_1\exp[-2\lambda(\alpha - a_0)]\} \times F(k_1,k_2,k_3).$$

With the infinite series of images converging uniformly, we can develop $F(k_1,k_2,k_3)$ in powers of k_i (i = 1–3). According to a commonly used method [7], the expression obtained is introduced in (2) and the integral and summation signs are interchanged. Note that the potential of interest in medium 1 can be identified with that created in an infinite homogeneous medium (of dielectric constant ε_1) by an initial charge and a quadruple infinity of images. The amplitude and position of these charges are shown in Table I.

The potential analytic expression for medium 1 is cumbersome, however, we remark inside the dielectric medium that there is an only charge, the initial charge. That is the singularity. Then, we can write for a line charge situated in $\{\rho = 0,\ u = \alpha\}$

$$\phi(\rho,u,\alpha) = \frac{1}{2\pi\varepsilon_1}\log\{\sqrt{\rho^2 + (u-\alpha)^2}\} + H(\rho,u,\alpha)$$

where $H(\rho,u,\alpha)$ is a continuous function. That is very important for computational purposes.

The calculations justify the passage of a two-dimensional case to a three-dimensional case by a mere change in the formulas of $(1/2\pi\varepsilon)\log\rho$ by $(1/4\pi\varepsilon)\cdot(1/r)$. The fictitious charges are evidently found exterior to domain 1, hence allowing the verification of Gaussian theorem.

In the case of a triplate configuration with an inhomogeneous dielectric medium [Fig. 1(a)], the following conditions must be

TABLE I

AMPLITUDE	POSITION
$+1$	α
$(-1)^{i+1} C_i^j C_j^k k_1^{(i-j+k)} \cdot k_2^{(1+j)} \cdot k_3^{(i-k)}$	$-\alpha+2a_1+2b_1(i-j+k)+2b_2(i-k)$
$(-1)^{i+1} C_i^j C_j^k k_1^{(i-j+k)} \cdot k_2^{j} \cdot k_3^{(1+i-k)}$	$-\alpha+2a_2+2b_1(i-j+k)+2b_2(i-k)$
$(-1)^{i+1} C_i^j C_j^k k_1^{(1+i-j+k)} \cdot k_2^{(1+j)} \cdot k_3^{(i-k)}$	$+\alpha+2b_1(1+i-j+k)+2b_2(i-k)$
$(-1)^{i+1} C_i^j C_j^k k_1^{(1+i-j+k)} \cdot k_2^{j} \cdot k_3^{(1+i-k)}$	$+\alpha+2b_1(1+i-j+k)+2b_2(1+i-k)$
k_1	$-\alpha+2a_0$
$(-1)^{i+1} C_i^j C_j^k k_1^{(1+i-j+k)} \cdot k_2^{(1+j)} \cdot k_3^{(i-k)}$	$+\alpha-2b_1(1+i-j+k)-2b_2(i-k)$
$(-1)^{i+1} C_i^j C_j^k k_1^{(2+i-j+k)} \cdot k_2^{(1+j)} \cdot k_3^{(i-k)}$	$-\alpha+2a_0-2b_1(1+i-j+k)-2b_2(i-k)$
$(-1)^{i+1} C_i^j C_j^k k_1^{(1+i-j+k)} \cdot k_2^{j} \cdot k_3^{(1+i-k)}$	$+\alpha-2b_1(1+i-j+k)-2b_2(1+i-k)$
$(-1)^{i+1} C_i^j C_j^k k_1^{(2+i-j+k)} \cdot k_2^{j} \cdot k_3^{(1+i-k)}$	$-\alpha+2a_0-2b_1(1+i-j+k)-2b_2(1+i-k)$

$i = 0, \infty$
$j = 0, i$
$k = 0, j$

$$C_i^j = \frac{i!}{j!\,(i-j)!}$$

$$k_n = \frac{\varepsilon_n - \varepsilon_{n-1}}{\varepsilon_n + \varepsilon_{n-1}}, \quad n = 1, 2, 3.$$

TABLE II

AMPLITUDE	POSITION
$+1$	h
$+k_1$	$-h$
$+k_1^{2i+1}$	$-h-2hi$
$+k_1^{2i}$	$+h-2hi$
$+k_1^{2i}$	$+h+2hi$
$+k_1^{2i-1}$	$-h+2hi$

imposed upon the general configuration (Fig. 2) in order to take into account the conductors:

$$\varepsilon_0 \longrightarrow \infty \qquad \varepsilon_3 \longrightarrow \infty.$$

This implies that $k_1 = 1$ and $k_3 = 1$; the expressions (16) given by Coen can then be found.

Without having to go into details, we would like to mention the practical problems which are solved (in using the results shown in Table I).

In a three-dimensional case, we studied the influence of an upper ground plane and a medium of several dielectrics on the elements of an equivalent circuit for discontinuities. The curves obtained recover exactly those given in (3).

In the two-dimensional case, this generalized method of images allows the propagation in structures shown in Fig. 1(b)–(d) to be treated [4]. Moreover, for coplanar conductors on a dielectric substrate, the formulas of Table I can be considerably simplified.

As an hypothesis, let $\varepsilon_2 = \varepsilon_3$ ($k_3 = 0$). The configuration [Fig. 1(e)] can then be easily treated. The amplitudes of charges decrease extremely rapidly as shown in Table II.

The geometrical configuration [Fig. 1(d)] allows a configuration identical to the preceding one to be treated but sandwiched.

We can conclude that depending upon the chosen integral representation, this method allows the two- or three-dimensional case to be treated by avoiding the singularity of the point source. This method is very interesting in solving open or semiopen problems. Knowing Green's function in a four dielectric medium, a general program valid for most transmission lines used in hyperfrequency microelectronics has been realized.

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The Locus of Points of Constant VSWR When Renormalized to a Different Characteristic Impedance

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Abstract—**The locus of points of constant VSWR with respect to an impedance Z_1, when measured with equipment of characteristic impedance Z_0, other than Z_1, is found to be a circle that is easily constructed on a Smith chart. The use of a transparent overlay with various such circles converts any network analyzer with a Smith chart display to read swept VSWR values to the new impedance.**

In cases where the VSWR of a device is specified with respect to an impedance Z_1 other than the characteristic impedance of the measuring equipment Z_0 (50 Ω), as is often the case with TV antenna equipment or cables, the VSWR has to be calculated from complex impedance measurements on a point-by-point basis. Thus the advantages of having a network analyzer with a sweep oscillator and Smith chart display are lost because the normal constant VSWR circles would refer to Z_0 rather than Z_1. This letter shows that the locus of points of constant VSWR to Z_1 are again circles, and that they can be very easily constructed, enabling the use of a transparent overlay and thus regaining the original network analyzer advantages.

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